

Evasion and prediction IV

Fragments of constant prediction

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February 8, 2008

Abstract

Say that a function $\pi : n^{<\omega} \rightarrow n$ (henceforth called a predictor) k -constantly predicts a real $x \in n^\omega$ if for almost all intervals I of length k , there is $i \in I$ such that $x(i) = \pi(x \restriction i)$. We study the k -constant prediction number $\mathfrak{p}_n^{\text{const}}(k)$, that is, the size of the least family of predictors needed to k -constantly predict all reals, for different values of n and k , and investigate their relationship.

*Supported by Grant-in-Aid for Scientific Research (C)(2)12640124, Japan Society for the Promotion of Science

Introduction

This work is about evasion and prediction, a combinatorial concept originally introduced by Blass when studying set-theoretic aspects of the Specker phenomenon in abelian group theory [Bl1]. The motivation for our investigation came from a (still open) question of Kamo, as well as from an argument in a proof by the first author. Let us explain this in some detail.

For our purposes, let $n \leq \omega$ and call a function $\pi : n^{<\omega} \rightarrow n$ a *predictor*. Say π *k-constantly predicts* a real $x \in n^\omega$ if for almost all intervals I of length k , there is $i \in I$ such that $x(i) = \pi(x \upharpoonright i)$. In case π *k-constantly predicts* x for some k , say that π *constantly predicts* x . The *constant prediction number* $\mathfrak{v}_n^{\text{const}}$, introduced by Kamo in [Ka1], is the smallest size of a set of predictors Π such that every $x \in n^\omega$ is constantly predicted by some $\pi \in \Pi$. Kamo [Ka1] showed that $\mathfrak{v}_\omega^{\text{const}}$ may be larger than all the $\mathfrak{v}_n^{\text{const}}$ where $n \in \omega$. He asked

Question. (Kamo [Ka2]) *Is $\mathfrak{v}_2^{\text{const}} = \mathfrak{v}_n^{\text{const}}$ for all $n \in \omega$.*

Some time ago, the first author answered another question of Kamo's by showing that $\mathfrak{b} \leq \mathfrak{v}_2^{\text{const}}$ where \mathfrak{b} is the unbounding number [Br]. Now, the standard approach to such a result would have been to show that, given a model M of *ZFC* such that there is a dominating real f over M , there must be a real which is not constantly predicted by any predictor from M . This, however, is far from being true. In fact, one needs a sequence of $2^k - 1$ models M_i and dominating reals f_i over M_i belonging to M_{i+1} to be able to construct a real which is not *k-constantly* predicted by any predictor from M_0 , and this result is optimal (see [Br] for details). This means *k-constant* prediction gets easier in a strong sense the larger k gets, and one can expect interesting results when investigating the cardinal invariants which can be distilled out of this phenomenon.

Accordingly, let us define the *k-constant prediction number* $\mathfrak{v}_n^{\text{const}}(k)$ to be the size of the smallest set of predictors Π such that every $x \in n^\omega$ is *k-constantly* predicted by some $\pi \in \Pi$. Interestingly enough, Kamo's question cited above has a positive answer when relativized to the new situation. Namely, we shall show in Section 1 that $\mathfrak{v}_2^{\text{const}}(k) = \mathfrak{v}_n^{\text{const}}(k)$ for all $k, n < \omega$ (see 1.4). Moreover, for $k < \ell$, one may well have $\mathfrak{v}_2^{\text{const}}(\ell) < \mathfrak{v}_2^{\text{const}}(k)$ (Theorem 2.1). Any hope to use Theorem 1.4 as an intermediate step to answer Kamo's question is dashed, however, by Theorem 2.2 which says that $\mathfrak{v}_2^{\text{const}}$ may be strictly smaller than the minimum of all $\mathfrak{v}_2^{\text{const}}(k)$'s.

In Section 3, we dualize Theorem 2.1 to a consistency result about evasion numbers and establish a connection between those and Martin's axiom for $\sigma - k$ -linked partial orders (see Theorem 3.7).

We keep our notation fairly standard. For basics concerning the cardinal invariants considered here, as well as the forcing techniques, see [BJ] and [Bl2].

The results in this paper were obtained in September 2000 during and shortly after the second author's visit to Kobe. The results in Sections 1 and 2 are due to the second author. The remainder is the first author's work.

1 The ZFC-results

Temporarily say that $\pi : n^{<\omega} \rightarrow n$ *weakly k -constantly predicts* $x \in n^\omega$ if for almost all m there is $i < k$ such that $\pi(x \upharpoonright mk + i) = x(mk + i)$. This notion is obviously weaker than k -constant prediction. It is often more convenient, however. We shall see soon that in terms of cardinal invariants the two notions are the same.

Put $G = \{\bar{g} = \langle g_i; i < k \rangle; g_i : n^k \rightarrow 2\}$.

Theorem 1.1 *There are functions $\bar{\pi} = \langle \pi^{\bar{g},j}; (\bar{g}, j) \in G \times k \rangle \mapsto \psi_{\bar{\pi}}$ (where $\pi^{\bar{g},j} : 2^{<\omega} \rightarrow 2$ and $\psi_{\bar{\pi}} : n^{<\omega} \rightarrow n$) and $y \mapsto \langle y^{\bar{g},j}; (\bar{g}, j) \in G \times k \rangle$ (where $y \in n^\omega$ and $y^{\bar{g},j} \in 2^\omega$) such that if $\pi^{\bar{g},j}$ weakly k -constantly predicts $y^{\bar{g},j}$ for all pairs (\bar{g}, j) , then $\psi_{\bar{\pi}}$ k -constantly predicts y .*

Proof. Given $y \in n^\omega$, define $y^{\bar{g},j}$ by

$$y^{\bar{g},j}(mk + i) = g_i(y \upharpoonright [mk + j, (m + 1)k + j)).$$

Also, for $\sigma \in n^{<\omega}$, say $|\sigma| = m_0k + j$, define $\sigma^{\bar{g},j}$ by

$$\sigma^{\bar{g},j}(mk + i) = g_i(\sigma \upharpoonright [mk + j, (m + 1)k + j))$$

for all $m < m_0$. So $|\sigma^{\bar{g},j}| = m_0k$.

Given $\bar{\pi} = \langle \pi^{\bar{g},j}; (\bar{g}, j) \in G \times k \rangle$, a sequence of predictors for the space 2^ω , and $\sigma \in n^{<\omega}$, say $|\sigma| = mk + j$, put

$$A_\sigma^k = \{\tau \supset \sigma; |\tau| = |\sigma| + k \text{ and } \forall \bar{g} \exists i (\tau^{\bar{g},j}(mk + i) = \pi^{\bar{g},j}(\tau^{\bar{g},j} \upharpoonright mk + i))\}.$$

For $i < k$, define $A_\sigma^i = \{\tau \supset \sigma; \tau \in A_{\sigma \upharpoonright |\sigma| - k + i}^k\}$. So, if $\tau \in A_\sigma^i$, $|\tau| = |\sigma| + i$.

Claim 1.2 $|A_\sigma^k| < 2^k$ for all σ .

Proof. Assume that, for some σ , we have $|A_\sigma^k| \geq 2^k$. List $\{\tau_\ell; \ell < 2^k\} \subseteq A_\sigma^k$ and list $2^k = \{\sigma_\ell; \ell < 2^k\}$. Fix m and j such that $|\sigma| = mk + j$. Define $g_i(\tau_\ell \upharpoonright [mk + j, (m+1)k + j]) = \sigma_\ell(i)$ and consider $\bar{g} = \langle g_i; i < k \rangle$. Then $\tau_\ell^{\bar{g},j} \upharpoonright [mk, (m+1)k] = \sigma_\ell$. This is a contradiction to the definition of A_σ^k for it would mean $\pi^{\bar{g},j}$ cannot predict correctly all $\tau_\ell^{\bar{g},j}$ somewhere in the interval $[mk, (m+1)k]$. \square

For $\sigma \in n^{<\omega}$ define $\psi_\pi(\sigma)$ as follows. First let $i \leq k$ be minimal such that $|A_\sigma^i| < 2^i$. Such i exists by the claim. Then let $\psi_\pi(\sigma)$ be any ℓ such that $A_{\sigma \upharpoonright \langle \ell \rangle}^{i-1}$ is of maximal size.

To see that this works, let $y \in n^\omega$. Let $\pi^{\bar{g},j}$ be predictors such that for all \bar{g}, j and almost all m , there is i such that $y^{\bar{g},j}(mk+i) = \pi^{\bar{g},j}(y^{\bar{g},j} \upharpoonright mk+i)$. Fix m_0 such that for all $m \geq m_0$ and all \bar{g}, j , there is i such that $y^{\bar{g},j}(mk+i) = \pi^{\bar{g},j}(y^{\bar{g},j} \upharpoonright mk+i)$. Let $mk+j \in \omega$ with $m \geq m_0$. Thus $y \upharpoonright mk+j+i \in A_{y \upharpoonright mk+j}^i$ for all $i \leq k$. We need to find $i < k$ such that $\psi_\pi(y \upharpoonright mk+j+i) = y(mk+j+i)$. To this end simply note that if i is such that $\psi_\pi(y \upharpoonright mk+j+i) \neq y(mk+j+i)$, then, by definition of ψ_π ,

$$|A_{y \upharpoonright mk+j+i+1}^{\ell_i-1}| \leq \frac{|A_{y \upharpoonright mk+j+i}^{\ell_i}|}{2}$$

where ℓ_i is minimal with $|A_{y \upharpoonright mk+j+i}^{\ell_i}| < 2^{\ell_i}$. This means in particular $|A_{y \upharpoonright mk+j+i+1}^{\ell_i-1}| < 2^{\ell_i-1}$. A fortiori, $\ell_{i+1} \leq \ell_i - 1$. Since $\ell_0 \leq k$, this entails that if we had $\psi_\pi(y \upharpoonright mk+j+i) \neq y(mk+j+i)$ for all $i < k$, we would get $\ell_i = 0$ for some $i \leq k$. Thus $|A_{y \upharpoonright mk+j+i}^0| < 2^0 = 1$. So $A_{y \upharpoonright mk+j+i}^0 = \emptyset$. However $y \upharpoonright mk+j+i \in A_{y \upharpoonright mk+j+i}^0$, a contradiction. This completes the proof of the theorem. \square

Define the k -constant evasion number $\mathfrak{e}_n^{\text{const}}(k)$ to be the dual of $\mathfrak{v}_n^{\text{const}}(k)$, namely the size of the smallest set of functions $F \subseteq n^\omega$ such that for every predictor π there is $x \in F$ which is not k -constantly predicted by π . Similarly, define the constant evasion number $\mathfrak{e}_n^{\text{const}}$.

Let $\bar{\mathfrak{v}}_n^{\text{const}}(k)$ denote the size of the least family Π of predictors $\pi : n^{<\omega} \rightarrow n$ such that every $y \in n^\omega$ is weakly k -constantly predicted by a member of Π . Dually, $\bar{\mathfrak{e}}_n^{\text{const}}(k)$ is the size of the least family $F \subseteq n^\omega$ such that no predictor $\pi : n^{<\omega} \rightarrow n$ weakly k -constantly predicts all members of F . The above theorem entails

Corollary 1.3 $\mathfrak{v}_n^{\text{const}}(k) \leq \bar{\mathfrak{v}}_2^{\text{const}}(k)$. Dually, $\mathfrak{e}_n^{\text{const}}(k) \geq \bar{\mathfrak{e}}_2^{\text{const}}(k)$.

Proof. Let Π be a family of predictors in 2^ω weakly k -constantly predicting all functions. Put $\Psi = \{\psi_{\bar{\pi}}; \bar{\pi} = \langle \pi^{\bar{g},j}; (\bar{g},j) \in G \times k \rangle \in \Pi^{<\omega}\}$. By the theorem, every $y \in n^\omega$ is k -constantly predicted by a member of Ψ . This shows $\mathfrak{v}_n^{\text{const}}(k) \leq \bar{\mathfrak{v}}_2^{\text{const}}(k)$.

Next let $F \subseteq n^\omega$ be a family of functions such that no predictor k -constantly predicts all of F . Let $Y = \{y^{\bar{g},j}; (\bar{g},j) \in G \times k \text{ and } y \in F\} \subseteq 2^\omega$. Assume $\pi : 2^{<\omega} \rightarrow 2$ weakly k -constantly predicts all members of Y . Then $\psi_{\bar{\pi}}$ k -constantly predicts all members of F , where we put $\bar{\pi} = \langle \pi^{\bar{g},j}; (\bar{g},j) \in G \times k \rangle$ with $\pi^{\bar{g},j} = \pi$ for all $(\bar{g},j) \in G \times k$, a contradiction. \square

Since the other inequalities are trivial, we get

Theorem 1.4 $\bar{\mathfrak{v}}_n^{\text{const}}(k) = \mathfrak{v}_n^{\text{const}}(k) = \mathfrak{v}_2^{\text{const}}(k)$ for all n . Dually, $\bar{\mathfrak{e}}_n^{\text{const}}(k) = \mathfrak{e}_n^{\text{const}}(k) = \mathfrak{e}_2^{\text{const}}(k)$ for all n .

A fortiori, we also get $\min\{\mathfrak{v}_n^{\text{const}}(k); k \in \omega\} = \min\{\mathfrak{v}_2^{\text{const}}(k); k \in \omega\}$ and $\sup\{\mathfrak{e}_n^{\text{const}}(k); k \in \omega\} = \sup\{\mathfrak{e}_2^{\text{const}}(k); k \in \omega\}$ for all n .

2 Prediction and relatives of Sacks forcing

For $2 \leq k < \omega$, define k -ary Sacks forcing \mathbb{S}^k to be the set of all subtrees $T \subseteq k^{<\omega}$ such that below each node $s \in T$, there is $t \supset s$ whose k immediate successor nodes $t \hat{\ } \langle i \rangle$ ($i < k$) all belong to T . \mathbb{S}^k is ordered by inclusion. Obviously \mathbb{S}^2 is nothing but standard Sacks forcing \mathbb{S} .

Iterating \mathbb{S}^k ω_2 many times with countable support over a model for CH yields a model where $\mathfrak{v}_2^{\text{const}}(\ell)$ is large if $2^\ell \leq k$ and small otherwise. This has been observed independently around the same time by Kada [Kd2]. However, one can get better consistency results by using large countable support products instead. The following is in the spirit of [GSh].

Theorem 2.1 Assume CH . Let $2 \leq k_1 < \dots < k_{n-1}$. Also let κ_i , $i \leq n$, be cardinals with $\kappa_i^\omega = \kappa_i$ and $\kappa_n < \dots < \kappa_0$. Then there is a generic extension satisfying $\mathfrak{v}_2^{\text{const}} = \min\{\mathfrak{v}_2^{\text{const}}(k); k \in \omega\} = \mathfrak{v}_2^{\text{const}}(k_{n-1} + 1) = \kappa_n$, $\mathfrak{v}_2^{\text{const}}(k_i) = \mathfrak{v}_2^{\text{const}}(k_{i-1} + 1) = \kappa_i$ for $0 < i < n$ and $\mathfrak{c} = \kappa_0$.

Proof. We force with the countable support product $\mathbb{P} = \prod_{\alpha < \kappa_0} \mathbb{Q}_\alpha$ where

- \mathbb{Q}_α is Sacks forcing \mathbb{S}_α for $\kappa_1 \leq \alpha < \kappa_0$,
- \mathbb{Q}_α is 2^{k_i} -ary Sacks forcing $\mathbb{S}_\alpha^{2^{k_i}}$ for $0 < i < n$ and $\kappa_{i+1} \leq \alpha < \kappa_i$, and
- \mathbb{Q}_α is $\mathbb{S}_\alpha^{\ell_\alpha}$ where $|\{\alpha; \ell = \ell_\alpha\}| = \kappa_n$ for all ℓ , for $\alpha < \kappa_n$.

By CH , \mathbb{P} preserves cardinals and cofinalities. $\mathfrak{c} = \kappa_0$ is also immediate.

Note that if $X \subseteq 2^\omega$ and $|X| < \kappa_i$, then there is $A \subseteq \kappa_0$ of size $< \kappa_i$ such that $X \in V[G_A]$, the generic extension by conditions with support contained in A , i.e. via the ordering $\prod_{\alpha \in A} \mathbb{Q}_\alpha$. So there is $\alpha \in (\kappa_i \setminus \kappa_{i+1}) \setminus A$. Clearly the generic real added by $\mathbb{Q}_\alpha = \mathbb{S}_\alpha^{2^{k_i}}$ is not k_i -constantly predicted by any predictor from $V[G_A]$. This shows $\mathfrak{v}_2^{\text{const}}(k_i) \geq \kappa_i$. A similar argument shows $\mathfrak{v}_2^{\text{const}} \geq \kappa_n$.

So it remains to see that $\mathfrak{v}_2^{\text{const}}(k_{i_0-1} + 1) \leq \kappa_{i_0}$ for $0 < i_0 \leq n$. Put $\ell = k_{i_0-1} + 1$. Let \dot{f} be a \mathbb{P} -name for a function in 2^ω . By a standard fusion argument we can recursively construct

- a strictly increasing sequence m_j , $j \in \omega$,
- $A \subseteq \kappa_0$ countable,
- $\langle D_\alpha; \alpha \in A \rangle$, a partition of ω into countable sets,
- a condition $p = \langle p_\alpha; \alpha \in A \rangle \in \mathbb{P}$, and
- a tree $T \subseteq 2^{<\omega}$

such that

- if $\sigma \in T \cap 2^{m_j}$, $j \in D_\alpha$, and $\alpha \in \kappa_i \setminus \kappa_{i+1}$ ($i < n$), then $|\{\tau \in T \cap 2^{m_{j+1}}; \sigma \subseteq \tau\}| = 2^{k_i}$ where we put $k_0 = 1$,
- $p \Vdash \dot{f} \in [T]$, and
- whenever $q \leq p$ where $q = \langle q_\beta; \beta \in B \rangle$ with $A \subseteq B$, $\sigma \in T \cap 2^{m_j}$, and $j \in D_\alpha$ are such that $q \Vdash \sigma \subseteq \dot{f}$, then there are $r_\alpha \leq q_\alpha$ and $\tau \in T \cap 2^{m_{j+1}}$ with $\tau \supseteq \sigma$, such that $r \Vdash \tau \subseteq \dot{f}$ where $r = \langle r_\beta; \beta \in B \rangle$ with $r_\beta = q_\beta$ for $\beta \neq \alpha$.

Now let $G_{\kappa_{i_0}}$ be $\prod_{\alpha < \kappa_{i_0}} \mathbb{Q}_\alpha$ -generic with $p \restriction \kappa_{i_0} \in G_{\kappa_{i_0}}$. By (c) above, there is, in $V[G_{\kappa_{i_0}}]$, a tree $S \subseteq T$ such that for all $\alpha \in A \cap \kappa_{i_0}$, $j \in D_\alpha$ and $\sigma \in S \cap 2^{m_j}$, there is a unique $\tau \in S \cap 2^{m_{j+1}}$ extending σ , and such that \dot{f} is

forced to be a branch of S by the remainder of the forcing below p . By (a), we also have that for all $\alpha \in A \setminus \kappa_{i_0}$, $j \in D_\alpha$ and $\sigma \in S \cap 2^{m_j}$, there are at most $2^{k_{i_0}-1}$ many $\tau \in S \cap 2^{m_{j+1}}$ extending σ . This means we can recursively construct a predictor $\pi \in V[G_{\kappa_{i_0}}]$ which ℓ -constantly predicts all branches of S . A fortiori, \dot{f} is forced to be predicted by π by the remainder of the forcing below p . On the other hand, $V[G_{\kappa_{i_0}}]$ satisfies $\mathfrak{c} = \kappa_{i_0}$ so that there are a total number of κ_{i_0} many predictors in $V[G_{\kappa_{i_0}}]$, and they ℓ -constantly predict all reals of the final extension. This completes the argument. \square

It is easy to see that in models obtained by such product constructions, $\mathfrak{v}_2^{\text{const}} = \min\{\mathfrak{v}_2^{\text{const}}(k); k \in \omega\}$ must always hold. To distinguish between these two cardinals, we must turn once again to a countable support iteration.

Theorem 2.2 *Assume CH. There is a generic extension satisfying $\mathfrak{v}_2^{\text{const}} = \aleph_1 < \min\{\mathfrak{v}_2^{\text{const}}(k); k \in \omega\} = \mathfrak{c} = \aleph_2$.*

Proof. Let $\langle k_\alpha; \alpha < \omega_2 \rangle$ be a sequence of natural numbers ≥ 2 in which each k appears ω_2 often and such that in each limit ordinal, the set of α with $k_\alpha = 2$ is cofinal.

We perform a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha; \alpha < \omega_2 \rangle$ such that

$$\Vdash_\alpha \text{“}\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{S}}^{k_\alpha}, \text{ that is } k_\alpha\text{-ary Sacks forcing.”}$$

By CH, \mathbb{P}_{ω_2} preserves cardinals and cofinalities. As in the previous proof, we see $\mathfrak{v}_2^{\text{const}}(k) = \mathfrak{c} = \aleph_2$ for all k . We are left with showing that $\mathfrak{v}_2^{\text{const}} = \aleph_1$.

Let \dot{f} be a \mathbb{P}_{ω_2} -name for a function in 2^ω . Notice given any $p_0 \in \mathbb{P}_{\omega_2}$, we can find $p \leq p_0$ and $\alpha < \omega_2$ such that

$$p \Vdash \dot{f} \in V[\dot{G}_\alpha] \setminus \bigcup_{\beta < \alpha} V[\dot{G}_\beta].$$

First consider the case α is a successor ordinal, say $\alpha = \beta + 1$. Let ℓ be such that $2^\ell > k_\beta$. The following is the main point.

Main Claim 2.3 *There are $q \leq p$ and a predictor $\pi \in V$ such that*

$$q \Vdash \text{“}\pi \text{ } \ell\text{-constantly predicts } \dot{f} \text{.”}$$

Proof. We construct recursively

- $A \subseteq \alpha$ countable,
- $\langle D_\gamma; \gamma \in A \rangle$, a partition of ω into countable sets,
- finite partial functions $a_j : A \rightarrow \omega$, $j \in \omega$,
- conditions $p_j \in \mathbb{P}_\alpha$, $j \in \omega$,
- a strictly increasing sequence m_j , $j \in \omega$,
- a tree $T \subseteq 2^{<\omega}$, and
- a predictor $\pi : 2^{<\omega} \rightarrow 2$

such that

- (a) $\beta \in A$,
- (b) $a_0 = \emptyset$,
- (c) if $j \in D_\gamma$, then $\text{dom}(a_{j+1}) = \text{dom}(a_j) \cup \{\gamma\}$; in case $\gamma \notin \text{dom}(a_j)$, we have $a_{j+1}(\gamma) = 0$, otherwise $a_{j+1}(\gamma) = a_j(\gamma) + 1$; $a_{j+1}(\delta) = a_j(\delta)$ for $\delta \neq \gamma$,
- (d) $p_0 = p$,
- (e) $p_{j+1} \leq p_j$; furthermore for all $\gamma \in \text{dom}(a_{j+1})$,
 $p_{j+1} \restriction \gamma \Vdash_\gamma p_{j+1}(\gamma) \leq_{a_{j+1}(\gamma)} p_j(\gamma)$,
- (f) $\bigcup_j \text{dom}(p_j) = \bigcup_j \text{dom}(a_j) = A$,
- (g) if $\sigma \in T \cap 2^{m_j}$, $j \in D_\gamma$, then $|\{\tau \in T \cap 2^{m_{j+1}}; \sigma \subseteq \tau\}| = k_\gamma$,
- (h) for each $\sigma \in T \cap 2^{m_j}$, there is $p_j^\sigma \leq p_j$ which forces $\sigma \subseteq \dot{f}$; furthermore $p_j \Vdash \dot{f} \restriction m_j \in T \cap 2^{m_j}$, and
- (i) π ℓ -constantly predicts all branches of T .

Most of this is standard. There is, however, one trick involved, and we describe the construction. For $j = 0$, there is nothing to do. So assume we arrived at stage j , and we are supposed to produce the required objects for $j + 1$. This proceeds by recursion on $\sigma \in T \cap 2^{m_j}$. Since the recursion is straightforward, we confine ourselves to describing a single step.

Fix $\sigma \in T \cap 2^{m_j}$. Let γ be such that $j \in D_\gamma$. Without loss $\gamma < \beta$ (the case $\gamma = \beta$ being easier). Consider p_j^σ . Step momentarily into $V[G_\beta]$ with $p_j^\sigma \restriction \beta \in G_\beta$. Then $p_j^\sigma(\beta) \Vdash_{\mathbb{Q}_\beta} \sigma \subseteq \dot{f}$. Since \dot{f} is forced not to be in $V[G_\beta]$, we can find $m^\sigma \in \omega$, pairwise incompatible $r_i^\sigma \leq p_j^\sigma(\beta)$, and distinct $\tau_i^\sigma \in 2^{m^\sigma}$ where $i < k_\gamma$ extending σ such that $r_i^\sigma \Vdash_{\mathbb{Q}_\beta} \tau_i^\sigma \subseteq \dot{f}$. As \mathbb{Q}_β is k_β -ary Sacks forcing, we may do this in such a way that the predictor π can be extended to ℓ -constantly predict all τ_i^σ .

Back in V , by extending the condition p_j^σ if necessary, we may without loss assume that it decides m^σ and the τ_i^σ . We therefore have the extension of π which ℓ -constantly predicts all τ_i^σ already in the ground model V . We may also suppose that $p_j^\sigma \restriction \gamma$ decides the stem of $p_j^\sigma(\gamma)$, say $p_j^\sigma \restriction \gamma \Vdash_\gamma \text{stem}(p_j^\sigma(\gamma)) = t$. For $i < k_\gamma$ define $p_{j+1}^{\tau_i^\sigma}$ such that

- $p_{j+1}^{\tau_i^\sigma} \restriction \gamma = p_j^\sigma \restriction \gamma$, $p_{j+1}^{\tau_i^\sigma} \restriction [\gamma + 1, \beta) = p_j^\sigma \restriction [\gamma + 1, \beta)$,
- $p_{j+1}^{\tau_i^\sigma} \restriction \gamma \Vdash_\gamma p_{j+1}^{\tau_i^\sigma}(\gamma) = (p_j^\sigma(\gamma))_{t \restriction \langle i \rangle}$,
- $p_{j+1}^{\tau_i^\sigma} \restriction \beta \Vdash_\beta p_{j+1}^{\tau_i^\sigma}(\beta) = \dot{r}_i^\sigma$.

Doing this (in a recursive construction) for all $\sigma \in T \cap 2^{m_j}$ and increasing m^σ if necessary, we may assume there is m_{j+1} with $m_{j+1} = m^\sigma$ for all σ . Finally p_{j+1} is the least upper bound of all the $p_{j+1}^{\tau_i^\sigma}$.

This completes the construction. By (c), (e), and (f), the sequence of p_j 's has a lower bound $q \in \mathbb{P}_\alpha$. By (d), $q \leq p$. By (h), $q \Vdash \dot{f} \in [T]$ which means that (i) entails $q \Vdash$ “ \dot{f} is ℓ -constantly predicted by π ,” as required. \square

Now let α be a limit ordinal. Using a similar argument and the fact that below α , \mathbb{Q}_β is cofinally often Sacks forcing, we see

Claim 2.4 *There are $q \leq p$ and a predictor $\pi \in V$ such that*

$$q \Vdash \text{“}\pi \text{ 2-constantly predicts } \dot{f} \text{.”}$$

This completes the proof of the theorem. \square

3 Evasion and fragments of $MA(\sigma\text{--}k\text{--}linked)$

Let $k \geq 2$. Recall that a partial order \mathbb{P} is said to be $\sigma - k$ -linked if it can be written as a countable union of sets P_n such that each P_n is k -linked, that is, any k many elements from P_n have a common extension. Clearly every σ -centered forcing is $\sigma - k$ -linked for all k , and a $\sigma - k$ -linked p.o. is also $\sigma - (k - 1)$ -linked. Random forcing is an example of a p.o. which is $\sigma - k$ -linked for all k , yet not σ -centered. A p.o. with the former property shall be called $\sigma - \infty$ -linked henceforth. We shall deal with p.o.'s which arise naturally in connection with constant prediction and which are $\sigma - (k - 1)$ -linked but not $\sigma - k$ -linked for some k . Let $\mathfrak{m}(\sigma - k\text{--}linked)$ denote the least cardinal κ such that for some $\sigma - k$ -linked p.o. \mathbb{P} , Martin's axiom MA_κ fails for \mathbb{P} .

Lemma 3.1 *Let \mathbb{P} be $\sigma - 2^k$ -linked, and assume $\dot{\phi}$ is a \mathbb{P} -name for a function $\bigcup_i 2^{ik} \rightarrow 2^k$. Then there is a countable set Ψ of functions $\bigcup_i 2^{ik} \rightarrow 2^k$ such that whenever $g \in 2^\omega$ is such that for all $\psi \in \Psi$ there are infinitely many i with $\psi(g \restriction ik) = g \restriction [ik, (i + 1)k)$, then*

$$\Vdash \text{“there are infinitely many } i \text{ with } \dot{\phi}(g \restriction ik) = g \restriction [ik, (i + 1)k). \text{”}$$

Proof. Assume $\mathbb{P} = \bigcup_n P_n$ where each P_n is 2^k -linked. Define $\psi_n : \bigcup_i 2^{ik} \rightarrow 2^k$ such that, for each $\sigma \in 2^{ik}$, $\psi_n(\sigma)$ is a τ such that no $p \in P_n$ forces $\dot{\phi}(\sigma) \neq \tau$. (Such a τ clearly exists. For otherwise, for each $\tau \in 2^k$ we could find $p_\tau \in P_n$ forcing $\dot{\phi}(\sigma) \neq \tau$. Since P_n is 2^k -linked, the p_τ would have a common extension which would force $\dot{\phi}(\sigma) \notin 2^k$, a contradiction.) Let $\Psi = \{\psi_n; n \in \omega\}$.

Now choose $g \in 2^\omega$ such that for all $\psi \in \Psi$ there are infinitely many i with $\psi(g \restriction ik) = g \restriction [ik, (i + 1)k)$. Fix i_0 and $p \in \mathbb{P}$. There is n such that $p \in P_n$. We can find $i \geq i_0$ such that $\psi_n(g \restriction ik) = g \restriction [ik, (i + 1)k)$. By definition of ψ_n , there is $q \leq p$ such that $q \Vdash \dot{\phi}(g \restriction ik) = \psi_n(g \restriction ik)$. Thus $q \Vdash \dot{\phi}(g \restriction ik) = g \restriction [ik, (i + 1)k)$, as required. \square

Lemma 3.2 *Let $\langle \mathbb{P}_n, \dot{\mathbb{Q}}_n; n \in \omega \rangle$ be a finite support iteration, and assume $\dot{\phi}$ is a \mathbb{P}_ω -name for a function $\bigcup_i 2^{ik} \rightarrow 2^k$. Also assume for each n and each \mathbb{P}_n -name $\dot{\phi}_n$ for a function $\bigcup_i 2^{ik} \rightarrow 2^k$, there is a countable set Ψ_n of functions $\bigcup_i 2^{ik} \rightarrow 2^k$ such that $\forall g \in 2^\omega$, if $\forall \psi \in \Psi_n \exists^\infty i (\psi(g \restriction ik) = g \restriction [ik, (i + 1)k))$, then*

$$\Vdash_n \text{“} \exists^\infty i (\dot{\phi}_n(g \restriction ik) = g \restriction [ik, (i + 1)k). \text{”}$$

Then there is a countable set Ψ of functions $\bigcup_i 2^{ik} \rightarrow 2^k$ such that $\forall g \in 2^\omega$, if $\forall \psi \in \Psi \exists^\infty i (\psi(g \upharpoonright ik) = g \upharpoonright [ik, (i+1)k))$, then

$$\Vdash_\omega \text{“}\exists^\infty i (\dot{\phi}(g \upharpoonright ik) = g \upharpoonright [ik, (i+1)k))\text{”}$$

Proof. This is a standard argument which we leave to the reader. \square

Lemma 3.3 *Let \mathbb{P} be a p.o. of size κ , and assume $\dot{\phi}$ is a \mathbb{P} -name for a function $\bigcup_i 2^{ik} \rightarrow 2^k$. Then there is a set Ψ of size κ of functions $\bigcup_i 2^{ik} \rightarrow 2^k$ such that $\forall g \in 2^\omega$, if $\forall \psi \in \Psi \exists^\infty i (\psi(g \upharpoonright ik) = g \upharpoonright [ik, (i+1)k))$, then*

$$\Vdash_\omega \text{“}\exists^\infty i (\dot{\phi}(g \upharpoonright ik) = g \upharpoonright [ik, (i+1)k))\text{”}$$

Proof. This is well-known and trivial. \square

Using the first two of these three lemmata we see that if we iterate $\sigma - 2^k$ -linked forcing over a model V containing a family $\mathcal{F} \subseteq 2^\omega$ such that

- (\star) for all countable sets Ψ of functions $\bigcup_i 2^{ik} \rightarrow 2^k$ there is $g \in \mathcal{F}$ with $\exists^\infty i (\psi(g \upharpoonright ik) = g \upharpoonright [ik, (i+1)k))$,

then \mathcal{F} still satisfies (\star) in the final extension. We also have

Lemma 3.4 *If \mathcal{F} satisfies (\star), then $\mathfrak{e}_2^{\text{const}}(k) \leq |\mathcal{F}|$.*

Proof. Simply note \mathcal{F} is a witness for $\mathfrak{e}_2^{\text{const}}(k)$. For given a predictor $\pi : 2^{<\omega} \rightarrow 2$, define $\phi : \bigcup_i 2^{ik} \rightarrow 2^k$ by $\phi(\sigma) =$ the unique $\tau \in 2^k$ such that π predicts $\sigma \hat{\ } \tau$ incorrectly on the whole interval $[ik, (i+1)k)$ where $|\sigma| = ik$. If $g \in \mathcal{F}$ is such that $\exists^\infty i (\phi(g \upharpoonright ik) = g \upharpoonright [ik, (i+1)k))$, then π does not k -constantly predict g . \square

Let $2 \leq k$. The partial order \mathbb{P}^k for adjoining a generic predictor k -constantly predicting all ground model reals is defined as follows. Conditions are triples (ℓ, σ, F) such that $\ell \in \omega$, $\sigma : 2^{<\omega} \rightarrow 2$ is a finite partial function, and $F \subseteq 2^\omega$ is finite, and such that the following requirements are met:

- $\text{dom}(\sigma) = 2^{\leq \ell}$,
- $f \upharpoonright \ell \neq g \upharpoonright \ell$ for all $f \neq g$ belonging to F ,
- $\sigma(f \upharpoonright \ell) = f(\ell)$ for all $f \in F$.

The order is given by: $(m, \tau, G) \leq (\ell, \sigma, F)$ if and only if $m \geq \ell$, $\tau \supseteq \sigma$, $G \supseteq F$, and for all $f \in F$ and all intervals $I \subseteq (\ell, m)$ of length k there is $i \in I$ with $\tau(f \upharpoonright i) = f(i)$. This is a variation of a p.o. originally introduced in [Br]. It has been considered as well by Kada [Kd1], who also obtained the following lemma.

Lemma 3.5 \mathbb{P}^k is $\sigma - (2^k - 1)$ -linked.

Proof. Simply adapt the argument from [Br, Lemma 3.2], or see [Kd1, Proposition 3.3]. \square

Corollary 3.6 (Kada [Kd1, Corollary 3.5]) $\mathfrak{m}(\sigma - (2^k - 1) - \text{linked}) \leq \mathfrak{e}_2^{\text{const}}(k)$.

We are ready to prove a result which is dual to Theorem 2.1.

Theorem 3.7 Let $\langle \kappa_k; 2 \leq k \in \omega \rangle$ be a sequence of uncountable regular cardinals with $\kappa_k \leq \kappa_{k+1}$. Also assume $\lambda = \lambda^{<\lambda}$ is above the κ_k . Then there is a generic extension satisfying $\mathfrak{e}_2^{\text{const}}(k) = \kappa_k$ for all k and $\mathfrak{c} = \lambda$. We may also get $\mathfrak{m}(\sigma - (2^k - 1) - \text{linked}) = \kappa_k$ for all k .

Proof. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha; \alpha < \lambda \rangle$ be a finite support iteration of ccc forcing such that each factor $\dot{\mathbb{Q}}_\alpha$ is forced to be a $\sigma - (2^k - 1)$ -linked forcing notion of size less than κ_k for some $k \geq 2$. Also guarantee we take care of all such forcing notions by a book-keeping argument. Then $\mathfrak{m}(\sigma - (2^k - 1) - \text{linked}) \geq \kappa_k$ is straightforward. In view of Corollary 3.6 it suffices to prove $\mathfrak{e}_2^{\text{const}}(k) \leq \kappa_k$ for all k . So fix k . Note that in stage κ_k of the iteration we adjoined a family \mathcal{F} of size κ_k satisfying (\star) above with *countable* replaced by *less than κ_k* . Show by induction on the remainder of the iteration that \mathcal{F} continues to satisfy this version of (\star) . The limit step is taken care of by Lemma 3.2. For the successor step, in case $\dot{\mathbb{Q}}_\alpha$ is $\sigma - 2^\ell$ -linked for some $\ell \geq k$, use Lemma 3.1, and in case it is not $\sigma - 2^k$ -linked (and thus of size less than κ_k), use Lemma 3.3. By Lemma 3.4, $\mathfrak{e}_2^{\text{const}}(k) \leq \kappa_k$ follows. \square

By somewhat changing the above proof, we can dualize Kamo's $CON(\mathfrak{v}_2^{\text{const}} > \text{cof}(\mathcal{N}))$ (and thus answer a question of his, see [Ka2]), and reprove his result as well.

Theorem 3.8 (a) $\mathfrak{e}_2^{\text{const}} < \text{add}(\mathcal{N})$ is consistent; in fact, given $\kappa < \lambda = \lambda^{<\kappa}$ regular uncountable, there is a p.o. \mathbb{P} forcing $\mathfrak{e}_2^{\text{const}} = \kappa$ and $\text{add}(\mathcal{N}) = \mathfrak{c} = \lambda$.

- (b) (Kamo, [Ka1]) $\mathfrak{v}_2^{\text{const}} > \text{cof}(\mathcal{N})$ is consistent; in fact, given κ regular uncountable and $\lambda = \lambda^\omega > \kappa$, there is a p.o. \mathbb{P} forcing $\mathfrak{v}_2^{\text{const}} = \mathfrak{c} = \lambda$ and $\text{cof}(\mathcal{N}) = \kappa$.

Proof. (a) Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha; \alpha < \lambda \rangle$ be a finite support iteration of ccc forcing such that

- for even α , $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha$ is amoeba forcing,
- for odd α , $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha$ is a subforcing of some \mathbb{P}^k of size less than κ .

Guarantee that we go through all such subforcings by a book-keeping argument. Then $\mathfrak{e}_2^{\text{const}} \geq \kappa$ is straightforward, as is $\text{add}(\mathcal{N}) = \mathfrak{c} = \lambda$. Now note that amoeba forcing is σ - ∞ -linked (like random forcing). Therefore we can apply Lemmata 3.1, 3.2, and 3.3 for all k simultaneously, and see that there is a family \mathcal{F} of size κ which satisfies the appropriate modified version of (\star) (such a family is adjoined after the first κ stages of the iteration).

(b) First add λ many Cohen reals. Then make a κ -stage finite support iteration of amoeba forcing. Again, $\text{cof}(\mathcal{N}) = \kappa$ is clear. $\mathfrak{v}_2^{\text{const}} = \mathfrak{c} = \lambda$ follows from Lemmata 3.1 and 3.2 using standard arguments. \square

One can even strengthen Theorem 3.7 in the following way. Say a p.o. \mathbb{P} satisfies property K_k if for all uncountable $X \subseteq \mathbb{P}$ there is $Y \subseteq X$ uncountable such that any k many elements from Y have a common extension. Property K_k is a weaker relative of σ - k -linkedness. Let $\mathfrak{m}(K_k)$ denote the least cardinal κ such that MA_κ fails for property K_k p.o.'s.

Lemma 3.9 *Assume CH. \mathbb{P}^k does not have property K_{2^k} . In fact no property K_{2^k} p.o. adds a predictor which k -constantly predicts all ground model reals.*

Proof. List all predictors as $\{\pi_\alpha; \alpha < \omega_1\}$. Choose reals $f_\alpha \in 2^\omega$ such that π_α does not k -constantly predict f_β for $\beta \geq \alpha$. Let $X = \{f_\alpha; \alpha < \omega_1\}$.

Let \mathbb{P} be property K_{2^k} . Also let $\dot{\pi}$ be a \mathbb{P} -name for a predictor. Assume there are conditions $p_\alpha \in \mathbb{P}$ such that $p_\alpha \Vdash$ “ $\dot{\pi}$ k -constantly predicts f_α from m_α onwards.” Without loss $m_\alpha = m$ for all α , and any 2^k many p_α have a common extension. Let $T \subseteq 2^{<\omega}$ be the tree of initial segments of members of X . Given $\sigma \in T$ with $|\sigma| \geq m$, let $A_\sigma^k = \{\tau \in T; \sigma \subset \tau \text{ and } |\tau| = |\sigma| + k\}$. Note that if $|A_\sigma^k| < 2^k$ for all such σ , then we could construct a predictor π k -constantly predicting all of X past m as in the proof of

Theorem 1.1. So there is $\sigma \in T$ with $|A_\sigma^k| = 2^k$. Find $\alpha_0, \dots, \alpha_{2^k-1}$ such that $A_\sigma^k = \{f_{\alpha_i} \mid |\sigma| + k; i < 2^k\}$ and notice that a common extension of the p_{α_i} forces a contradiction. \square

Note that some assumption is necessary for the above result for MA_{\aleph_1} implies all p.o.'s have property K_k for all k . We now get

Theorem 3.10 *Assume CH. Let $2 \leq k < \omega$. Then there is a generic extension satisfying $\mathfrak{e}_2^{\text{const}}(k) = \aleph_1$ and $\mathfrak{m}(K_{2^k}) = \aleph_2$.*

Proof. Use the lemma and the folklore fact that the iteration of property K_ℓ p.o.'s is property K_ℓ . \square

Since we saw in Corollary 3.6 that $\mathfrak{e}_2^{\text{const}}(k) \geq \mathfrak{m}(\sigma - (2^k - 1) - \text{linked})$, one may ask, on the other hand, whether $\mathfrak{e}_2^{\text{const}}(k) > \mathfrak{m}(\sigma - (2^k - 1) - \text{linked})$ is consistent. This, however, is easy, for the forcing \mathbb{P}^k is Suslin ccc [BJ] while it is well-known that iterating Suslin ccc forcing keeps numbers like $\mathfrak{m}(\sigma - (2^k - 1) - \text{linked})$ small (it even keeps the splitting number \mathfrak{s} small).

We close this section with a few questions. We have no dual result for Theorem 2.2 so far.

Question 3.11 *Is $\mathfrak{e}_2^{\text{const}} > \sup\{\mathfrak{e}_2^{\text{const}}(k); k < \omega\}$ consistent?*

Question 3.12 *Can $\mathfrak{e}_2^{\text{const}}$ have countable cofinality?*

By Theorem 3.7, either of these two questions must have a positive answer. In fact, in view of the proof of Theorem 3.8, $\mathfrak{e}_2^{\text{const}}$ must be

- either $\max\{\kappa_k; k \in \omega\}$ (in case the set has a max),
- or $\sup\{\kappa_k; k \in \omega\}$ or its successor (in case the set has no max)

in the model of Theorem 3.7.

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